直交曲線座標系における粘性散逸関数について

デカルト座標系(x,y,z)において，散逸関数は次式で与えられる．

$$\Phi=\frac{\mu}{2}\left(e_{ik}e_{ik}-\frac{4}{3}\Theta^2\right),\quad \left(e_{ik}=\frac{\partial u_k}{\partial x_i}+\frac{\partial u_i}{\partial x_k}=\vec{\nabla}\vec{u}+{}^t\left(\vec{\nabla}\vec{u}\right),\quad \Theta=\frac{\partial u_k}{\partial x_k}\right) \tag{1}$$

まず，デカルト座標系で展開を試みる．

$$e_{ik}e_{ik}=\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)=\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}+2\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_i}+\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_i}=2\left(\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}+\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_i}\right)$$

$$\frac{1}{2}e_{ik}e_{ik}=\frac{\partial u_1}{\partial x_k}\frac{\partial u_1}{\partial x_k}+\frac{\partial u_1}{\partial x_k}\frac{\partial u_k}{\partial x_1}+\frac{\partial u_2}{\partial x_k}\frac{\partial u_2}{\partial x_k}+\frac{\partial u_2}{\partial x_k}\frac{\partial u_k}{\partial x_2}+\frac{\partial u_3}{\partial x_k}\frac{\partial u_3}{\partial x_k}+\frac{\partial u_3}{\partial x_k}\frac{\partial u_k}{\partial x_3}$$

$$\begin{aligned}
&=\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_2}+\frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_3}+\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1}\\
&+\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_2}+\frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_3}+\frac{\partial u_2}{\partial x_1}\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_2}+\frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2}\\
&+\frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_1}+\frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\frac{\partial u_3}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_3}\frac{\partial u_3}{\partial x_3}
\end{aligned}$$

$$\begin{aligned}
&=2\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_2}+\frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_3}+2\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}+2\frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1}\\
&+\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_1}+2\frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_2}+\frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_3}+2\frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2}\\
&+\frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_1}+\frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_2}+2\frac{\partial u_3}{\partial x_3}\frac{\partial u_3}{\partial x_3}
\end{aligned}$$

$$\begin{aligned}
&=2\left(\frac{\partial u_1}{\partial x_1}\right)^2+2\left(\frac{\partial u_2}{\partial x_2}\right)^2+2\left(\frac{\partial u_3}{\partial x_3}\right)^2+\left(\frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_2}+2\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}+\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_1}\right)\\
&+\left(\frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_3}+2\frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2}+\frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_2}\right)+\left(\frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_1}+2\frac{\partial u_3}{\partial x_1}\frac{\partial u_1}{\partial x_3}+\frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_3}\right)
\end{aligned}$$

$$\therefore\quad \frac{1}{2}e_{ik}e_{ik}=2\left(\frac{\partial u_1}{\partial x_1}\right)^2+2\left(\frac{\partial u_2}{\partial x_2}\right)^2+2\left(\frac{\partial u_3}{\partial x_3}\right)^2+\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)^2+\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)^2+\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}\right)^2$$

$$\Phi=\mu\left[\begin{aligned}&2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+2\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2\\&-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2\end{aligned}\right] \tag{2}$$

次に円筒座標系(r,θ,z)の場合について展開する.

$$\Phi=\frac{\mu}{2}\left(\left\{\vec{\nabla}\otimes\vec{u}+{}^{t}\left(\vec{\nabla}\otimes\vec{u}\right)\right\}^{2}-\frac{4}{3}\Theta^{2}\right) \tag{3}$$

$$\Theta=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z} \tag{4}$$

$$\begin{aligned}\vec{\nabla}\otimes\vec{u}+{}^{t}\left(\vec{\nabla}\otimes\vec{u}\right)=&\vec{e}_r\otimes\vec{e}_r\left(2\frac{\partial u_r}{\partial r}\right)+\vec{e}_r\otimes\vec{e}_\theta\left(\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right)+\vec{e}_r\otimes\vec{e}_z\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)\\&+\vec{e}_\theta\otimes\vec{e}_r\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right)+\vec{e}_\theta\otimes\vec{e}_\theta\left(\frac{2}{r}\frac{\partial u_\theta}{\partial\theta}+2\frac{u_r}{r}\right)+\vec{e}_\theta\otimes\vec{e}_z\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta}+\frac{\partial u_\theta}{\partial z}\right)\\&+\vec{e}_z\otimes\vec{e}_r\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right)+\vec{e}_z\otimes\vec{e}_\theta\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\theta}\right)+\vec{e}_z\otimes\vec{e}_z\left(2\frac{\partial u_z}{\partial z}\right)\end{aligned}$$

$$\begin{aligned}\left\{\vec{\nabla}\otimes\vec{u}+{}^{t}\left(\vec{\nabla}\otimes\vec{u}\right)\right\}^{2}&=e_{rr}{}^{2}+e_{\theta\theta}{}^{2}+e_{zz}{}^{2}+2e_{r\theta}{}^{2}+2e_{\theta z}{}^{2}+2e_{zr}{}^{2}\\&=\left(2\frac{\partial u_r}{\partial r}\right)^{2}+\left(\frac{2}{r}\frac{\partial u_\theta}{\partial\theta}+2\frac{u_r}{r}\right)^{2}+\left(2\frac{\partial u_z}{\partial z}\right)^{2}\\&\quad+2\left(\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right)^{2}+2\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta}+\frac{\partial u_\theta}{\partial z}\right)^{2}+2\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right)^{2}\end{aligned}$$

$$\Phi=\mu\left[\begin{aligned}&2\left(\frac{\partial u_r}{\partial r}\right)^{2}+2\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)^{2}+2\left(\frac{\partial u_z}{\partial z}\right)^{2}+\left(\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right)^{2}\\&+\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta}+\frac{\partial u_\theta}{\partial z}\right)^{2}+\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right)^{2}-\frac{2}{3}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z}\right)^{2}\end{aligned}\right] \tag{5}$$

同様にして，球座標系(r,θ,ϕ)の場合には，散逸関数は次式となる.

$$\Phi=\mu\left[\begin{aligned}&2\left(\frac{\partial u_r}{\partial r}\right)^{2}+2\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)^{2}+2\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}+\frac{u_r}{r}+\frac{u_\theta\cot\theta}{r}\right)^{2}\\&+\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\phi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}\right)^{2}+\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi}+r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)\right)^{2}\\&+\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)^{2}\\&-\frac{2}{3}\left(\frac{1}{r^{2}}\frac{\partial}{\partial r}\left(r^{2}u_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right)^{2}\end{aligned}\right] \tag{6}$$

円筒座標系と球座標系とでは，それぞれr,θは別のものであることに注意.